

Dirac approach to constrained submanifolds in a double loop group: from WZNW to Poisson-Lie σ -model

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Abstract

We study the restriction to a family of second class constrained submanifolds in the cotangent bundle of a double Lie group, equipped with a 2-cocycle extended symplectic form, building the corresponding Dirac brackets. It is shown that, for a 2-cocycle vanishing on each isotropic subspaces of the associated Manin triple, the Dirac bracket contains no traces of the cocycle. We also investigate the restriction of the left translation action of the double Lie group on its cotangent bundle, where it fails in to be a symmetry a canonical transformation. However, the hamiltonian symmetry is restored on some special submanifolds. The main application is on loop groups, showing that a WZNW-type model on the double Lie group with a quadratic Hamilton function in the momentum maps associated with the left translation action on the cotangent bundle with the canonical symplectic form, restricts to a collective system on some special submanifolds. There, the lagrangian version coincides with so called Poisson-Lie σ -model.

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Many relevant physical systems are modeled on Lie groups, taking the corresponding cotangent bundles as their phase spaces. Among the finite dimensional examples are the rigid bodies and its generalizations [2] and, in infinite dimension, the sigma and WZNW models are outstanding field theories. Cotangent bundles are canonically symplectic manifolds and they have rich structures underlying symmetries which involves their Lie algebras and its dual spaces [1],[3],[15],[10]. In some special situations, these structures straightforwardly leads to integrability [18].

In the last decades, the dynamic of integrable systems becomes more involved with Lie groups: it turns out that most of them are deeply related to *Poisson-Lie groups*, that is, Lie groups supplied with a compatible Poisson structure [8]. A Poisson-Lie group has naturally associated a *dual* Poisson-Lie group, and the *double Lie group* build with this dual pair is, in some sense, a self dual structure with a lot of nice properties enriching the framework of integrability [8],[19],[13].

In reference [6], by regarding the cotangent bundle of this kind of double Lie group as a fibration on one of its factors, the Dirac method [7] was developed for dealing with the restriction to the fibers of a dynamical system on whole space. In fact, these fibers turns to be symplectic submanifolds of the cotangent bundle of the double Lie group equipped with the canonical symplectic form, turning the restriction to them in a second class constraint problem.

Cotangent bundles of Lie groups equipped with the *canonical symplectic structure* are not enough to encode all the plethora of systems modelled on Lie groups. For instance, the phase space of a sigma model with target space the group manifold G is the cotangent bundle T^*LG of the loop group LG with the canonical symplectic form ω_o , and the dynamics is determined by the election of the Hamilton function. On the other side, the WZNW model shares the same configuration space, but it can not be obtained from this phase space: no Hamilton function can be found driving to Hamilton equations equivalent to the WZNW ones. In fact, it was shown in ref. [9] that the addition of Wess-Zumino term, the *topological* term, to the action of the sigma model amounts to a modification of the canonical Poisson brackets. Its symplectic counterpart is exhaustively studied in references [11], where a cocycle extension of the canonical symplectic form ω_o is considered in combination with the Marsden-Weinstein reduction by stages procedure in order to recover the WZNW equation of motion [16].

In this work we adapt the scheme developed in [6] to the case where the initial phase space is the cotangent bundle T^*G of a double Poisson-Lie group $G = G_+G_-$ supplied with the symplectic form ω_c obtained by modifying the canonical one by adding a 2-cocycle $c : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R}$ on the Lie algebra \mathfrak{g} of the double Lie group G . So, Hamiltonian systems on this symplectic manifold are of the WZNW-type in the sense that their Lagrangian counterpart exhibits the topological WZ-term. We consider dynamical systems on the fiber of the fibration $T^*G \longrightarrow T^*G_-$ which are symplectic submanifolds of (T^*G, ω_c) , so they can be addressed via the Dirac method for second class constraints in analogous way as in [6]. Thus, we built the Dirac brackets to describe dynamical systems on these constrained submanifolds in terms of the algebra of function on the whole space T^*G . We investigate how the left action of the group G on itself, lifted to the cotangent bundle, restricts to the fibers and becomes in a symmetry of these phase subspaces for some special fibers. These scheme comes to be very useful when applied to the specific case of loop groups, where amazingly the Dirac brackets lose the cocycle contributions. We work out the restriction of a quadratic hamiltonian on T^*G which becomes collective on the same fibers where the left translation action turns to be a symmetry, recovering a Poisson-Lie σ -model on each of these fibers. All these facts bring the subject into the realm of Poisson-Lie T-duality [12], turning this machinery very useful for working on the hamiltonian approach to it [4],[5].

This work is organized as follows: we divided it in two main parts. In Part I we concentrate in developing the Dirac machinery and symmetry issues for the 2-cocycle extended

symplectic form on T^*G . Thus, in Section 1 we describe the phase space on a double Poisson-Lie Lie group and build the fibration of constrained submanifolds. In the Section 2 we adapt the scheme developed in [6] to the case where the canonical symplectic form is modified by adding a 2-cocycle. Section 3 is devoted to specialize the construction to loop groups. In Section 4 we discuss the left translation symmetry including the action by left translation of the centrally extended group. Section 5 introduces the Hamilton equations for the whole and the constrained spaces, and describes some properties of the collective dynamics. In Part II we develop the main application of the results of the Part I, mainly addressed to the loop group context. So, in Section 6 we address a hamiltonian model which restrict to a collective one. In Sections 7 and 8 we retrieve the Lagrange equations, introducing explicitly the loop groups stage, making contact with the so called Poisson-Lie σ -models. Finally, in the last Section some conclusions are summarized.

Part I

Phase spaces on double Lie groups and constrained systems

In this first part we study the fibration $\Psi : G \times \mathfrak{g}^* \longrightarrow G_- \times \mathfrak{g}_-^*$, for a double Lie group $G = G_+G_-$, as the phase space of systems constrained to the fibers $\Psi^{-1}(g_-, \eta_-)$. We adapt the Dirac's machinery developed in [6] to the framework of $G \times \mathfrak{g}^*$ equipped with the 2-cocycle extended symplectic form, pointing to the loop groups stage. We also address the left translation action of G on $G \times \mathfrak{g}^*$ and its restriction to $\Psi^{-1}(g_-, \eta_-)$, finding out the fibers on which it turns a phase space symmetry with Ad-equivariant momentum maps. Collective dynamics is then possible on some fibers, so we study its properties in the current framework.

1 The fibration $G \times \mathfrak{g}^* \longrightarrow G_- \times \mathfrak{g}_-^*$

Let us describe the framework for the main developments in this work (we follow the definitions and notations in [13]). Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be *Manin triple*, that means, $\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-$ are Lie algebras such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a vector space, and \mathfrak{g} is supplied with a ad-invariant nondegenerate symmetric bilinear form $(\cdot, \cdot)_{\mathfrak{g}}$ turning $\mathfrak{g}_+, \mathfrak{g}_-$ into isotropic subspaces. Thus, \mathfrak{g} is a *double Lie algebra* and $\mathfrak{g}_+, \mathfrak{g}_-$ are *Lie bialgebras*, a couple of dual Lie bialgebras. The associated connected simply connected Lie group G, G_+, G_- form a *double Lie group* $G = G_+G_-$, with G_+ and G_- being *Poisson-Lie groups*. Let us denote the corresponding projectors $\Pi_{G_{\pm}} : G \longrightarrow G_{\pm}$, $\Pi_{\mathfrak{g}_{\pm}} : \mathfrak{g} \longrightarrow \mathfrak{g}_{\pm}$ and $\Pi_{\mathfrak{g}_{\pm}^*} : \mathfrak{g}^* \longrightarrow \mathfrak{g}_{\pm}^*$, that for short we frequently denote as $g_{\pm} = \Pi_{G_{\pm}}g$, $X_{\pm} = \Pi_{\mathfrak{g}_{\pm}}X$ and $\eta_{\pm} = \Pi_{\mathfrak{g}_{\pm}^*}\eta$. The factorization of the elements like $g_-g_+ \in G$ is denoted as

$$g_+^{h_-} := \Pi_{G_+}(h_-g_+) \quad , \quad g_-^{h_+} := \Pi_{G_-}(h_+g_-)$$

Indeed, the maps

$$G_{\mp} \times G_{\pm} \longrightarrow G_{\pm} \quad / \quad (h_{\mp}, g_{\pm}) \mapsto \Pi_{G_{\pm}}(h_{\mp}g_{\pm}) = g_{\pm}^{h_{\mp}}$$

amounts to be crossed actions between the factors, the so called *dressing actions* [19],[13]. The infinitesimal generator of the dressing action of G_- on G_+ at the point $g_+ \in G_+$ gives

rise to the antihomomorphism of Lie algebras $X_- \in \mathfrak{g}_- \mapsto g_+^{X_-} \in T_{g_+}G_+$, such that, for $X_-, Y_- \in \mathfrak{g}_-$, $[g_+^{X_-}, g_+^{Y_-}] = -g_+^{[X_-, Y_-]_{\mathfrak{g}_-}}$.

Let ψ be the identification $\mathfrak{g} \longrightarrow \mathfrak{g}^*$ provided by the nondegenerate bilinear form, and $\bar{\psi}$ denote its inverse, then the crossed adjoint actions are

$$\begin{cases} \text{Ad}_{h_+^G} X_- = h_+^{-1} h_+^{X_-} + \bar{\psi} \left(\text{Ad}_{h_+^*}^* \psi(X_-) \right) \\ \text{Ad}_{h_-^G} X_+ = h_-^{X_+} h_-^{-1} + \bar{\psi} \left(\text{Ad}_{h_-^*}^* \psi(X_+) \right) \end{cases}$$

where Ad^* stands for the dual of the adjoint action of each group factor on the dual of its Lie algebra (it relates with the coadjoint action $\text{Ad}^\#$ as $\text{Ad}_{h_+}^\# := \text{Ad}_{h_+^*}^*$). This expression allows to write the infinitesimal generators as $g_+^{X_-} = g_+ \left(\Pi_{\mathfrak{g}_+} \text{Ad}_{g_+^G} X_- \right)$ and $g_-^{X_+} = \left(\Pi_{\mathfrak{g}_-} \text{Ad}_{g_-^G} X_+ \right) g_-$.

The starting point of our developments is the cotangent bundle of the double Lie group $G = G_+ G_-$, realized as $G \times \mathfrak{g}^*$ by using the left translation isomorphism, and the fibration defined by the surjective submersion

$$\begin{aligned} \Psi : G \times \mathfrak{g}^* &\longrightarrow G_- \times \mathfrak{g}_-^* \\ (g, \eta) &\longmapsto (g_-, \eta_-) \end{aligned}$$

Let us name $\mathcal{N}(g_-, \eta_-)$ the fiber on (g_-, η_-) , then

$$\mathcal{N}(g_-, \eta_-) := \Psi^{-1}(g_-, \eta_-) = \{(g_+ g_-, \eta_+ + \eta_-) / g_+ \in G_+, \eta_+ \in \mathfrak{g}_+^*\}$$

The differential Ψ_* of the map $\Psi : G \times \mathfrak{g}^* \longrightarrow G_- \times \mathfrak{g}_-^*$ can be obtained from

$$g^{-1} \dot{g} = \text{Ad}_{g_-^G}^G g_+^{-1} \dot{g}_+ + g_-^{-1} \dot{g}_-$$

then,

$$\Psi_*(gX, \xi)|_{(g, \eta)} = \left(\left(\Pi_{\mathfrak{g}_-} \text{Ad}_{g_-^G} X_+ \right) g_- + g_- X_-, \xi_- \right)_{(g_-, \eta_-)}$$

The kernel of Ψ_* coincides with $T\mathcal{N}(g_-, \eta_-)$, and it can be explicitly described as

$$\ker \Psi_*|_{(g, \eta)} = \left\{ \left(g_+ \left(\bar{\psi} \left(\text{Ad}_{g_-^*}^* \psi(X_+) \right) \right) g_-, \xi_+ \right) / (X_+, \xi_+) \in \mathfrak{g}_+ \oplus \mathfrak{g}_+^* \right\}$$

The Dirac method is build from the annihilator of $\ker \Psi_*|_{(g, \eta)}$, that is the pullback of the cotangent bundle of the base space $G_- \times \mathfrak{g}_-^*$.

2 Centrally extended symplectic structures and the Dirac method

In this section we adapt the Dirac bracket construction of ref. [6] to the case where canonical symplectic form on $G \times \mathfrak{g}^*$ is modified by adding a \mathbb{R} -valued 2-cocycle on \mathfrak{g} and we write the Dirac bracket on the submanifolds $\mathcal{N}_c(g_-, \eta_-)$.

Let $C : G \longrightarrow \mathfrak{g}^*$ be a coadjoint 1-cocycle, that is, for $g, h \in G$ it satisfy

$$C(gh) = \text{Ad}_{g^{-1}}^G C(h) + C(g) \quad , \quad \forall g, h \in G$$

By considering $\hat{c} = -dC|_e : \mathfrak{g} \longrightarrow \mathfrak{g}^*$, the 1-cocycle C defines the application $c : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R}$ given by

$$c(X, Y) := \langle \hat{c}(X), Y \rangle$$

Its easy to see that c is bilinear, antisymmetric and verifies the Jacobi identity. Then, c is a \mathbb{R} -valued 2-cocycle on \mathfrak{g} that satisfy the following condition

$$c(Ad_g X, Ad_g Y) = c(X, Y) + \langle C(g^{-1}), [X, Y] \rangle$$

So, let us consider the canonical symplectic form ω_o in T^*G . Then, by adding c one defines a new symplectic form on T^*G given by

$$\begin{aligned} & \langle \omega_c, (v, \rho) \otimes (w, \xi) \rangle_{(g, \eta)} \\ & : = -\langle \rho, g^{-1}w \rangle + \langle \xi, g^{-1}v \rangle + \langle \eta, [g^{-1}v, g^{-1}w] \rangle + c(vg^{-1}, wg^{-1}) \end{aligned} \quad (1)$$

for $(v, \rho), (w, \xi) \in T_{(g, \eta)}^*G$.

The hamiltonian vector field of a function $\mathcal{F} : G \times \mathfrak{g}^* \longrightarrow \mathbb{R}$ is

$$V_{\mathcal{F}}(g, \eta) = \left(g\delta\mathcal{F}, ad_{\delta\mathcal{F}}^*\eta - g\mathbf{d}\mathcal{F} + Ad_g^{G*}\hat{c}(Ad_g^G\delta\mathcal{F}) \right) \quad (2)$$

for $(g, \eta) \in G \times \mathfrak{g}^*$, and the associated Poisson bracket is

$$\{\mathcal{F}, \mathcal{G}\}_c(g, \eta) = \langle \delta\mathcal{G}, g\mathbf{d}\mathcal{F} \rangle - \langle \mathbf{d}\mathcal{G}, g\delta\mathcal{F} \rangle - \langle \eta + C(g^{-1}), [\delta\mathcal{F}, \delta\mathcal{G}] \rangle - c(\delta\mathcal{F}, \delta\mathcal{G})$$

for $\mathcal{F}, \mathcal{G} \in C^\infty(G \times \mathfrak{g}^*)$.

Let us now proceed to adapt the Dirac method to the restrictions to the fibers $\mathcal{N}(g_-, \eta_-)$ when the centrally extended symplectic form on T^*G is considered.

Proposition: $(\mathcal{N}(g_-, \eta_-), \tilde{\omega}_c)$, where $\tilde{\omega}_c$ is the restriction to $\mathcal{N}(g_-, \eta_-)$ of the centrally extended canonical symplectic form ω_c on $G \times \mathfrak{g}^*$, is a symplectic manifold.

Proof: Let us observe that the restriction of ω_c to the kernel of Ψ_* , is given by

$$\begin{aligned} & \left\langle \omega_c, \left(g_+ \left(\bar{\psi} \left(Ad_{g_-^{-1}}^* \psi(X_+) \right) \right) g_-, \xi_+ \right) \otimes \left(g_+ \left(\bar{\psi} \left(Ad_{g_-^{-1}}^* \psi(Y_+) \right) \right) g_-, \lambda_+ \right) \right\rangle_{(g, \eta)} \\ & = \left\langle \omega, \left(g_+ \left(\bar{\psi} \left(Ad_{g_-^{-1}}^* \psi(X_+) \right) \right) g_-, \xi_+ \right) \otimes \left(g_+ \left(\bar{\psi} \left(Ad_{g_-^{-1}}^* \psi(Y_+) \right) \right) g_-, \lambda_+ \right) \right\rangle_{(g, \eta)} \\ & \quad + c \left(g_+ \left(\bar{\psi} \left(Ad_{g_-^{-1}}^* \psi(X_+) \right) \right) g_-, g_+ \left(\bar{\psi} \left(Ad_{g_-^{-1}}^* \psi(Y_+) \right) \right) g_- \right) \end{aligned}$$

for $\left(g_+ \left(\bar{\psi} \left(Ad_{g_-^{-1}}^* \psi(X_+) \right) \right) g_-, \xi_+ \right), \left(g_+ \left(\bar{\psi} \left(Ad_{g_-^{-1}}^* \psi(Y_+) \right) \right) g_-, \lambda_+ \right) \in \ker \Psi_*|_{(g, \eta)}$. Then, is easy to see that there are not null vectors of ω_c on $T\Psi^{-1}(g_-, \eta_-) = \ker \Psi_*|_{(g, \eta)}$. ■

Corollary: $(T_{(g, \eta)}\mathcal{N}(g_-, \eta_-))^{\perp \omega_c} \cap T_{(g, \eta)}\mathcal{N}(g_-, \eta_-) = \{0\}$, then $\Psi^{-1}(g_-, \eta_-)$ is a second class constraint.

Here, $(T_{(g, \eta)}\mathcal{N}(g_-, \eta_-))^{\perp \omega_c}$ denotes the orthogonal complement in relation with the symplectic structure ω_c .

Now, we choose a basis for $T_{(g_-, \eta_-)}^*(G_- \times \mathfrak{g}_-^*) \cong \mathfrak{g}_-^* \oplus \mathfrak{g}_-$, where we used the left trivialization of $T_{(g_-, \eta_-)}^*G_-$. The basis $\{T_a\}$ of \mathfrak{g}_+ and the basis $\{T^a\}$ of \mathfrak{g}_- provides a set of linearly independent 1-forms on $G_- \times \mathfrak{g}_-^*$:

$$\begin{aligned} \alpha_a &= \left(L_{g_-^{-1}}^* \psi(T_a), 0 \right) \in T_{(g_-, \eta_-)}^*G_- \times \mathfrak{g}_-^* \\ \beta_a &= (0, T^a) \in T_{(g_-, \eta_-)}^*G_- \times \mathfrak{g}_-^* \end{aligned}$$

such that, for any $(v_-, \xi_-) \in T_{(g_-, \eta_-)} G_- \times \mathfrak{g}_-^*$

$$\begin{aligned}\langle \alpha_a, (v_-, \xi_-) \rangle_{(g_-, \eta_-)} &= \langle T_a, g_-^{-1} v_- \rangle_{\mathfrak{g}} \\ \langle \beta_a, (v_-, \xi_-) \rangle_{(g_-, \eta_-)} &= \langle \xi_-, T^a \rangle\end{aligned}$$

The set of pullbacks $\{\Psi^* \alpha_a, \Psi^* \beta^a\}_a$ is linearly independent, with $T\mathcal{N}(g_-, \eta_-)$ being its null distribution.

The associated hamiltonian vector fields $V_{\Psi^* \alpha_a}$ and $V_{\Psi^* \beta^a}$ are

$$\begin{cases} V_{\Psi^* \alpha_a}(g, \eta) = \left(0, -\text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}+} \text{Ad}_{g_-}^G T_a\right)_{(g, \eta)} \\ V_{\Psi^* \beta^a}(g, \eta) = \left(g T^a, \text{ad}_{T^a}^{\mathfrak{g}*} \eta - \lambda \text{Ad}_g^{G*} \hat{c} \left(\text{Ad}_g^G T^a\right)\right) \end{cases}$$

which allows to we calculate the Dirac matrix

$$C(g, \eta) = \begin{pmatrix} C_{\Psi^* \alpha_a, \Psi^* \alpha_b}(g, \eta) & C_{\Psi^* \alpha_a, \Psi^* \beta^b}(g, \eta) \\ -C_{\Psi^* \alpha_a, \Psi^* \beta^b}(g, \eta) & C_{\Psi^* \beta^a, \Psi^* \beta^b}(g, \eta) \end{pmatrix}$$

determined by the entries

$$C_{\Psi^* \alpha_a, \Psi^* \alpha_b}(g, \eta) = \langle \Psi^* \alpha_a, V_{\Psi^* \alpha_b} \rangle_{(g, \eta)} = \langle \Psi^* \alpha_a, (0, w_{\alpha_b}) \rangle_{(g, \eta)} = 0$$

$$C_{\Psi^* \alpha_a, \Psi^* \beta^b}(g, \eta) = \langle \Psi^* \alpha_a, V_{\Psi^* \beta^b} \rangle_{(g, \eta)} = \delta_a^b$$

$$C_{\Psi^* \beta^a, \Psi^* \beta^b}(g, \eta) = \langle \Psi^* \beta^a, V_{\Psi^* \beta^b} \rangle_{(g, \eta)} = \Omega_c^{ab}(g, \eta)$$

Here we wrote

$$\Omega_c^{ab}(g, \eta) := -\langle C(g^{-1}) + \eta, [T^a, T^b] \rangle - c(T^a, T^b)$$

The Dirac matrix is then

$$C(g, \eta) = \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ -I_{n \times n} & \Omega_c(g, \eta) \end{pmatrix}$$

Now, we are ready to built up the Dirac brackets: for any couple of function $\mathcal{F}, \mathcal{G} \in C^\infty(G \times \mathfrak{g}^*)$, the Dirac bracket gives the restriction of the Poisson bracket on $G \times \mathfrak{g}^*$ to the constrained submanifold $\mathcal{N}(g_-, \eta_-)$, and it is defined as

$$\begin{aligned}\{\mathcal{F}, \mathcal{G}\}_c^D(g, \eta) &= \{\mathcal{F}, \mathcal{G}\}_c(g, \eta) \\ &\quad + \langle \eta + C(g^{-1}), [\Pi_{\mathfrak{g}-} \{\mathcal{F}, \alpha\}, \Pi_{\mathfrak{g}-} \{\alpha, \mathcal{G}\}] \rangle_{(g, \eta)} \\ &\quad + \langle \{\mathcal{F}, \alpha\}, \{\beta, \mathcal{G}\} \rangle - \langle \{\mathcal{F}, \beta\}, \{\alpha, \mathcal{G}\} \rangle_{(g, \eta)} \\ &\quad + c(\Pi_{\mathfrak{g}-} \{\mathcal{F}, \alpha\}_c(g, \eta), \Pi_{\mathfrak{g}-} \{\alpha, \mathcal{G}\}_c(g, \eta))\end{aligned}$$

where

$$\Pi_{\mathfrak{g}-} \{\mathcal{F}, \alpha\} = \{\mathcal{F}, \alpha_a\}(g, \eta) T^a = -\text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}-} \text{Ad}_{g_-}^G \delta \mathcal{F}$$

$$\Pi_{\mathfrak{g}+} \{\mathcal{F}, \beta\}(g, \eta) = \Pi_{\mathfrak{g}+} (g \mathbf{d} \mathcal{F} - \text{ad}_{\delta \mathcal{F}}^{\mathfrak{g}*} (C(g^{-1}) + \eta) + c(T^a, \delta \mathcal{F}) T_a)$$

Thus, we have the following result.

Proposition: *The Dirac bracket on the submanifolds $\mathcal{N}_c(g_-, \eta_-)$ for any couple of function $\mathcal{F}, \mathcal{G} \in C^\infty(G \times \mathfrak{g}^*)$ is*

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\}_c^D(g, \eta) &= \left\langle g \mathbf{d}\mathcal{F}, \text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{G} \right\rangle - \left\langle g \mathbf{d}\mathcal{G}, \text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{F} \right\rangle \\ &\quad - \left\langle \eta, [\text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{F}, \text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{G}] \right\rangle \\ &\quad - \left\langle C(g_+^{-1}), [\Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{F}, \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{G}] \right\rangle \\ &\quad - c\left(\Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{F}, \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{G}\right) \end{aligned} \quad (3)$$

It is a nondegenerate bracket.

Remark I: *In the particular case $(g_-, \eta_-) = (e, 0)$ we recover the cotangent bundle of G_+ endowed with the Poisson structure*

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\}_c^D(g_+, \eta_+) &= \left\langle g \mathbf{d}\mathcal{F}, \Pi_{\mathfrak{g}_+} \delta \mathcal{G} \right\rangle - \left\langle g \mathbf{d}\mathcal{G}, \Pi_{\mathfrak{g}_+} \delta \mathcal{F} \right\rangle \\ &\quad - \left\langle \eta + C(g_+^{-1}), [\Pi_{\mathfrak{g}_+} \delta \mathcal{F}, \Pi_{\mathfrak{g}_+} \delta \mathcal{G}] \right\rangle \\ &\quad - c(\Pi_{\mathfrak{g}_+} \delta \mathcal{F}, \Pi_{\mathfrak{g}_+} \delta \mathcal{G}) \end{aligned}$$

Remark II: *Let us suppose that the restriction of the cocycle on G to G_\pm is such that*

$$C|_{G_\pm} : G_\pm \longrightarrow \mathfrak{g}_\mp^*$$

which in turn implies that

$$\hat{c}|_{\mathfrak{g}_\pm} : \mathfrak{g}_\pm \longrightarrow \mathfrak{g}_\mp^*$$

then, for any couple of function $\mathcal{F}, \mathcal{G} \in C^\infty(G \times \mathfrak{g}^)$, the Dirac bracket on $\mathcal{N}_c(g_-, \eta_-)$ of eq. (3) reduces to*

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\}_c^D(g, \eta) &= \left\langle g \mathbf{d}\mathcal{F}, \text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{G} \right\rangle - \left\langle g \mathbf{d}\mathcal{G}, \text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{F} \right\rangle \\ &\quad - \left\langle \eta, [\text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{F}, \text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{G}] \right\rangle \end{aligned} \quad (4)$$

where there are no traces of the cocycle. Moreover, at $(g_-, \eta_-) = (e, 0)$ it reduces to

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\}_c^D(g_+, \eta_-) &= \left\langle g \mathbf{d}\mathcal{F}, \Pi_{\mathfrak{g}_+} \delta \mathcal{G} \right\rangle - \left\langle g \mathbf{d}\mathcal{G}, \Pi_{\mathfrak{g}_+} \delta \mathcal{F} \right\rangle \\ &\quad - \left\langle \eta, [\Pi_{\mathfrak{g}_+} \delta \mathcal{F}, \Pi_{\mathfrak{g}_+} \delta \mathcal{G}] \right\rangle \end{aligned}$$

the restriction to the tangent bundle of G_+ equipped with the canonical symplectic form. The hamiltonian vector field in this case is

$$V_{\mathcal{H}}(g, \eta) = \left(g \left(\text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{H} \right), \text{Ad}_{g_-}^{G*} \Pi_{\mathfrak{g}_+}^* \text{Ad}_{g_-^{-1}}^{G*} \left(\text{ad}_{\text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{H}}^{\mathfrak{g}^*} \eta \right) \right) \quad (5)$$

3 Loop groups

Loop groups constitutes the main application of the above construction, also considering their central extension. Thus, let $G = LH$ denotes the set of maps from S^1 to the Lie group H , and $\mathfrak{g} = L\mathfrak{h}$ the maps of S^1 into the Lie algebra \mathfrak{h} of H . We assume the \mathfrak{h} is equipped with a nondegenerate Ad^H -invariant symmetric bilinear form $(\cdot, \cdot)_{\mathfrak{h}}$.

For $g \in G$, g' denotes the derivative in the loop parameter $s \in S^1$, and we write vg^{-1} and $g^{-1}v$ for the right and left translation of any vector field $v \in TG$. Frequently we will work with the dense subset $L\mathfrak{h}^* \subset (L\mathfrak{h})^*$ instead of $(L\mathfrak{h})^*$ itself, and we identify it with $L\mathfrak{h}$ through the map $\psi : L\mathfrak{h} \rightarrow L\mathfrak{h}^*$ provided by the bilinear form

$$(\cdot, \cdot)_{\mathfrak{g}} \equiv \frac{1}{2\pi} \int_{S^1} (\cdot, \cdot)_{\mathfrak{h}}$$

on \mathfrak{g} . In this framework, the two cocycle $c_k : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ is given by the bilinear form $\Gamma_k : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ [17],

$$c_k(X, Y) \equiv \Gamma_k(X, Y) = \frac{k}{2\pi} \int_{S^1} (X(s), Y'(s))_{\mathfrak{h}} ds$$

with $X(s), Y(s) \in \mathfrak{h}$. It is derived from the one cocycle $C_k : G \rightarrow \mathfrak{g}^*$,

$$C_k(l) = k\psi(l'l^{-1})$$

Observe that is an coadjoint cocycle

$$C_k(kl) = \text{Ad}_{k^{-1}}^{G^*} C_k(l) + C_k(k)$$

As above, we assume that $H = H_+H_-$ with H_+, H_- being Lie subgroups of H , and $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$, where $\mathfrak{h}_+, \mathfrak{h}_-$ are Lie subalgebras of \mathfrak{h} . Moreover, we assume that the subspaces $\mathfrak{h}_+, \mathfrak{h}_-$ are isotropic in relation with the bilinear form $(\cdot, \cdot)_{\mathfrak{h}}$. Then, the restriction of the bijection $\psi : \mathfrak{h} \rightarrow \mathfrak{h}^*$ to \mathfrak{h}_{\pm} provides the identification $\psi : \mathfrak{h}_{\pm} \rightarrow \mathfrak{h}_{\mp}^*$, and the restriction of the cocycle to the factors G_{\pm} is then the map $C_k : G_{\pm} \rightarrow \mathfrak{h}_{\mp}^*$. Moreover, the bilinear form $(\cdot, \cdot)_{\mathfrak{h}}$ and the 2 cocycle $c_k : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ restricted to \mathfrak{g}_{\pm} vanish, falling in the situation of *Remark II* of the previous section.

Thus, the Dirac bracket on the fiber $\mathcal{N}(g_-, \eta_-)$ coincides with (4). This is the framework for the developments in the second part of this work.

4 The left action of G on $G \times \mathfrak{g}^*$ and its restriction to $\mathcal{N}(g_-, \eta_-)$

This section is devoted to study the left action of G on $G \times \mathfrak{g}^*$ and how it restricts to the fibers $\mathcal{N}(g_-, \eta_-)$, analyzing the momentum maps borrowed from the phase space $G \times \mathfrak{g}^*$ supplied with the canonical symplectic structure.

The selected cocycle breaks explicitly the left action symmetry then it is no longer an endomorphism on (T^*G, ω_c) . However, since the cocycle contribution seems to disappear from the Dirac brackets, we wonder if that symmetry would be restored on some the fibers $\mathcal{N}(g_-, \eta_-)$. If it were the case, the infinitesimal generator X_{T^*G} would be related to the momentum functions associated with the canonical symplectic form, so we consider the momentum map associated to left translation on (T^*G, ω_o) , namely $J_B^L : T^*G \rightarrow \mathfrak{g}^*$ defined as

$$J_B^L(g, \eta) = \text{Ad}_{g^{-1}}^{G^*} \eta$$

It is worth to stress that it is not a momentum map for (T^*G, ω_c) . Despite of this fact, we shall consider the associated momentum function j_X^L , namely

$$j_X^L(g, \eta) = \langle \text{Ad}_{g^{-1}}^{G*} \eta, X \rangle = \langle \eta, \text{Ad}_{g^{-1}}^G X \rangle$$

satisfying $\iota_{X_{T^*G}} \omega_c = dj_X^L$, and construct the associated hamiltonian vector fields. In doing so, we need the differential of j_X^L ,

$$dj_X^L = \left(g^{-1} \text{ad}_{\text{Ad}_{g^{-1}}^G X}^{\mathfrak{g}*} \eta, \text{Ad}_{g^{-1}}^G X \right)$$

So, for an arbitrary function \mathcal{F} on $G \times \mathfrak{g}^*$, its Lie derivative along the hamiltonian vector field of j_X^L projected on the tangent space to $\mathcal{N}(g_-, \eta_-)$ is given by the Dirac bracket

$$\begin{aligned} \{\mathcal{F}, j_X^L\}_c^D(g, \eta) &= \langle g \mathbf{d}\mathcal{F}, \text{Ad}_{g^{-1}}^G \Pi_{\mathfrak{g}+} \text{Ad}_{g^{-1}}^G X \rangle \\ &\quad - \langle \text{ad}_{\text{Ad}_{g^{-1}}^G X}^{\mathfrak{g}*} \eta, \text{Ad}_{g^{-1}}^G \Pi_{\mathfrak{g}+} \text{Ad}_{g^{-1}}^G \delta \mathcal{F} \rangle \\ &\quad - \langle \eta, [\text{Ad}_{g^{-1}}^G \Pi_{\mathfrak{g}+} \text{Ad}_{g^{-1}}^G \delta \mathcal{F}, \text{Ad}_{g^{-1}}^G \Pi_{\mathfrak{g}+} \text{Ad}_{g^{-1}}^G X] \rangle \end{aligned}$$

so, the hamiltonian vector field is

$$V_{j_X^L}(g, \eta) = \left(g \text{Ad}_{g^{-1}}^G \Pi_{\mathfrak{g}+} \text{Ad}_{g^{-1}}^G X, -\text{Ad}_{g^{-1}}^{G*} \Pi_{\mathfrak{g}+}^* \text{Ad}_{g^{-1}}^{G*} \text{ad}_{\text{Ad}_{g^{-1}}^G \Pi_{\mathfrak{g}+} \text{Ad}_{g^{-1}}^G X}^{\mathfrak{g}*} \eta \right)$$

For the fiber on $(e, 0)$ and $X = X_+ \in \mathfrak{g}_+$ we get

$$V_{j_{X_+}^L}(g, \eta) = (X_+ g_+, 0)$$

that is just the infinitesimal generator of the left action of G_+ on T^*G_+ .

A test to see if the symmetry is restored on $\mathcal{N}(g_-, \eta_-)$ is to calculate the Dirac bracket of two momentum functions. If the result is that the momentum function close a Lie algebra under the Dirac bracket, then we have a Lie algebra morphism between the Lie algebra of the group and the Lie algebra of momentum functions, showing that the symmetry is symplectically realized on $\mathcal{N}(g_-, \eta_-)$.

Proposition: *Let η_- a character of \mathfrak{g}_- . Then, the map $\mathfrak{g} \longrightarrow C^\infty(\mathcal{N}(g_-, \eta_-))$ such that $X \longrightarrow j_X^L$ is a Lie algebra homomorphism in relation with the Dirac bracket (4). in $\mathcal{N}(g_-, \eta_-)$.*

Proof: The Dirac bracket between momentum functions is

$$\begin{aligned} \{j_X^L, j_Y^L\}_c^D(g, \eta) &= \langle \eta, [\text{Ad}_{g^{-1}}^G X, \text{Ad}_{g^{-1}}^G \Pi_{\mathfrak{g}+} \text{Ad}_{g^{-1}}^G Y] \rangle \\ &\quad + \langle \eta, [\text{Ad}_{g^{-1}}^G \Pi_{\mathfrak{g}+} \text{Ad}_{g^{-1}}^G X, \text{Ad}_{g^{-1}}^G \Pi_{\mathfrak{g}-} \text{Ad}_{g^{-1}}^G Y] \rangle \end{aligned}$$

Observe that if η_- is a character of \mathfrak{g}_- we can add a term

$$0 = \langle \eta, [\text{Ad}_{g^{-1}}^G \Pi_{\mathfrak{g}-} \text{Ad}_{g^{-1}}^G X, \text{Ad}_{g^{-1}}^G \Pi_{\mathfrak{g}-} \text{Ad}_{g^{-1}}^G Y] \rangle$$

so $\{j_X^L, j_Y^L\}_c^D$ turns in

$$\{j_X^L, j_Y^L\}_c^D(g, \eta) = \langle \eta, [\text{Ad}_{g^{-1}}^G X, \text{Ad}_{g^{-1}}^G Y] \rangle$$

that is equivalent to

$$\{j_{X_+}^L, j_Y^L\}_c^D(g, \eta) = j_{[X, Y]}^L(g, \eta)$$

indicating that the left invariance under the action of G_+ is restored on $\mathcal{N}(g_-, \eta_-)$ when η_- is a character of \mathfrak{g}_- . ■

4.1 The action of the centrally extended group G^\wedge

Now, we consider the action of the centrally extended loop group G^\wedge which is more frequently involved in the hamiltonian framework of loop groups with extended symplectic form concerning WZNW models, rather than the simple action of the group itself. So, it seems natural to work out the symmetries generated in this case by repeating the developments of the previous section.

Let \mathfrak{g}_c the *centrally extended* Lie algebra \mathfrak{g} defined by the cocycle $c : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}$, and \mathfrak{g}_c^* its dual algebra. The adjoint and coadjoint actions of G coincides with those of the *centrally extended* Lie group G^\wedge since the action of the extension factor on \mathfrak{g}_c is trivial. Then, because \mathbb{R} acts trivially, the adjoint action are

$$\text{Ad}_{(g,b)}^{G^\wedge}(X, a) := \text{Ad}_g^G(X, a)$$

Explicitly we have the formulas

$$\begin{cases} \text{Ad}_g^G(X, a) := \left(\text{Ad}_g^G X, a + \langle C(g^{-1}), X \rangle \right) \\ \text{Ad}_{g^{-1}}^{G^*}(\xi, b) := \left(\text{Ad}_{g^{-1}}^{G^*} \xi + bC(g), b \right) \\ \text{ad}_X^{\mathfrak{g}}(Y, a) := ([X, Y], c(X, Y)) \end{cases}$$

Let us now introduce the action by left translations of G^\wedge on $G \times \mathfrak{g}^*$ by defining the *centrally extended momentum map*

$$J_B^{L^\wedge}(g, \eta) := \text{Ad}_{g^{-1}}^{G^*}(\eta, 1) = \left(\text{Ad}_{g^{-1}}^{G^*} \eta + C(g), 1 \right) \quad (6)$$

The momentum function $j_X^{L^\wedge}$ associated with $J_B^{L^\wedge}$, namely

$$j_{(X,a)}^{L^\wedge}(g, \eta) = \left\langle \text{Ad}_{g^{-1}}^{G^*} \eta, (X, a) \right\rangle = j_X^L(g, \eta) + \langle C(g), X \rangle + a$$

then

$$dj_{(X,a)}^{L^\wedge} = \left(g^{-1} \left(\text{ad}_{\text{Ad}_{g^{-1}}^G X}^{\mathfrak{g}^*} \eta + \hat{c} \left(\text{Ad}_{g^{-1}}^G X \right) \right), \text{Ad}_{g^{-1}}^G X \right)$$

The centrally extended canonical symplectic form (1) gives rise to the hamiltonian vector field

$$V_{j_{(X,b)}^{L^\wedge}}(g, \eta) = \left(Xg, \text{ad}_{\text{Ad}_{g^{-1}}^G X}^{\mathfrak{g}^*} C(g^{-1}) \right)$$

and Poisson bracket between momentum functions is

$$\left\{ j_{(X,a)}^{L^\wedge}, j_{(Y,b)}^{L^\wedge} \right\}_c(g, \eta) = \left\langle dj_{(X,a)}^{L^\wedge}, V_{j_{(Y,b)}^{L^\wedge}} \right\rangle_{(g,\eta)}$$

and the explicit calculation gives

$$\left\{ j_{(X,a)}^{L^\wedge}, j_{(Y,b)}^{L^\wedge} \right\}_c(g, \eta) = j_{[(X,a),(Y,b)]}^{L^\wedge}(g, \eta) + \langle C(g), [X, Y] \rangle$$

which reflects the non invariance of the symplectic form ω_c .

Now, let us study the behavior of the hamiltonian vector fields $V_{j_{(X,a)}^{L^\wedge}} \equiv V_{(X,a)}^{\mathcal{N}}$ on the fibers $\mathcal{N}(g_-, \eta_-)$ by calculating the Dirac brackets of these momentum functions. To start with, let us consider the Dirac bracket with an arbitrary function \mathcal{F} ,

$$\begin{aligned} \left\{ \mathcal{F}, j_{(X,a)}^{L^\wedge} \right\}_c^D(g, \eta) &= \left\langle g \mathbf{d}\mathcal{F}, \text{Ad}_{g_-}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_+}^G X \right\rangle \\ &\quad - \left\langle \text{ad}_{\text{Ad}_{g_-}^G X}^{\mathfrak{g}^*} \eta + \hat{c} \left(\text{Ad}_{g_-}^G X \right), \text{Ad}_{g_-}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{F} \right\rangle \\ &\quad - \left\langle \eta, [\text{Ad}_{g_-}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta \mathcal{F}, \text{Ad}_{g_-}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_+}^G X] \right\rangle \end{aligned}$$

from where we get the hamiltonian vector field

$$\begin{aligned} V_{(X,a)}^{\mathcal{N}}(g, \eta) &= \left(g \text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_+^{-1}}^G X, \right. \\ &\quad \left. - \text{Ad}_{g_-}^{G*} \Pi_{\mathfrak{g}_+}^* \text{Ad}_{g_-^{-1}}^{G*} \left(\text{ad}_{\text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_-} \text{Ad}_{g_+^{-1}}^G X}^{\mathfrak{g}*} \eta + \hat{c} \left(\text{Ad}_{g_-^{-1}}^G X \right) \right) \right) \end{aligned} \quad (7)$$

In analogous way as in the proposition at the end of the previous section, it is easy to see the following result.

Proposition: *Let $(g_-, \eta_-) \in \ker C \times \text{Char}(\mathfrak{g}_-)$, then the map $\mathfrak{g}^\wedge \rightarrow C^\infty(\mathcal{N}(g_-, \eta_-))$ such that $(X, a) \rightarrow j_{(X,a)}^{L^\wedge}$ is a Lie algebra homomorphism in relation with the Dirac bracket (4) on $C^\infty(\mathcal{N}(g_-, \eta_-))$, namely*

$$\left\{ j_{(X,a)}^{L^\wedge}, j_{(Y,b)}^{L^\wedge} \right\}_c^D(g, \eta) = j_{[(X,a), (Y,b)]}^{L^\wedge}(g, \eta)$$

In terms of hamiltonian vector fields it means that

$$\left[V_{(X,a)}^{\mathcal{N}}, V_{(Y,b)}^{\mathcal{N}} \right] = -V_{[(X,a), (Y,b)]}^{\mathcal{N}}$$

so, the linear map $\mathfrak{g}^\wedge \rightarrow \mathfrak{X}(\mathcal{N}(g_-, \eta_-)) / (X, a) \rightarrow V_{(X,a)}^{\mathcal{N}}$ is a Lie algebra antihomomorphism showing that it is the infinitesimal generator of a well defined symplectic left action of G^\wedge on $\mathcal{N}(g_-, \eta_-)$.

In analogous way as in ref. [6] in the framework of (T^*G, ω_o) , this infinitesimal action of \mathfrak{g} on $\mathcal{N}(g_-, \eta_-)$ gives rise to a finite action of G^\wedge on $\mathcal{N}(g_-, \eta_-)$.

Proposition: *The vector field $V_{(X,a)}^{\mathcal{N}} \in \mathfrak{X}(\mathcal{N}(g_-, \eta_-))$, for $X \in \mathfrak{g}$ and η_- a character of \mathfrak{g}_- , is the infinitesimal generator associated with the action $\mathbf{d}: G^\wedge \times \mathcal{N}(g_-, \eta_-) \rightarrow \mathcal{N}(g_-, \eta_-)$ defined as*

$$\begin{aligned} &\mathbf{d}((h, a), (g, \eta)) \\ &= \left(g \text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} (g_+^{-1} h g_+) , \right. \\ &\quad \left. \text{Ad}_{g_-}^{G*} \Pi_{\mathfrak{g}_+}^* \left(\text{Ad}_{\Pi_{G_-}(g_+^{-1} h g_+)}^{G*} \text{Ad}_{g_-}^{G*} \eta + C \left((\Pi_{G_-}(g_+^{-1} h g_+))^{-1} \right) \right) \right) \end{aligned} \quad (8)$$

$$\forall (g, \eta) = (g_+ g_-, \eta_+ + \eta_-) \in \mathcal{N}(g_-, \eta_-).$$

Proof: It follows by straightforward calculation of the differential of this map. ■

Remark: *Observe that for $g_- = e$ and $\eta_- = 0$ it turns into*

$$\begin{aligned} &\mathbf{d}((h, a), (g, \eta)) \\ &= \left(g_+ \Pi_{\mathfrak{g}_+} (g_+^{-1} h g_+) , \right. \\ &\quad \left. \Pi_{\mathfrak{g}_+}^* \left(\text{Ad}_{\Pi_{G_-}(g_+^{-1} h g_+)}^{G*} \eta_+ + C \left((\Pi_{G_-}(g_+^{-1} h g_+))^{-1} \right) \right) \right) \end{aligned}$$

This action was introduced in [4],[5] as the fundamental ingredient underlying the hamiltonian Poisson Lie T-duality scheme.

In so far we have then show that, despite there are no left translation symmetry on the phase space (T^*G, ω_c) , it is restored on some particular fibers $\mathcal{N}(g_-, \eta_-)$ turning them into potentially interesting phase space for systems symmetric under the projection on $\mathcal{N}(g_-, \eta_-)$ of the left action of G^\wedge on T^*G .

5 The Hamilton equations and collective dynamics

In order to study dynamical systems with the kind of phase space described above, we explicitly write the Hamilton equations in the whole space T^*G and those on the constrained submanifolds $\mathcal{N}(g_-, \eta_-)$ produced by the Dirac brackets, both for a generic Hamilton function \mathcal{H} on T^*G .

The Hamilton equations on T^*G are determined by the hamiltonian vector field associated to a Hamilton function \mathcal{H} through the symplectic form (1), that was given in (2). So, defining $d\mathcal{H} = (\mathbf{d}\mathcal{H}, \delta\mathcal{H}) \in T_{(g,\eta)}^*(G \times \mathfrak{g}^*)$, these Hamilton equation are

$$\begin{cases} g^{-1}\dot{g} = \delta\mathcal{H} \\ \dot{\eta} = \text{ad}_{\delta\mathcal{H}}^* \eta - g\mathbf{d}\mathcal{H} + \text{Ad}_g^{G*} \hat{c} \left(\text{Ad}_g^G \delta\mathcal{H} \right) \end{cases}$$

From the hamiltonian vector field (5), which is tangent to $\mathcal{N}(g_-, \eta_-)$, we get the Hamilton equations on this phase space:

$$\begin{cases} g^{-1}\dot{g} = \text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta\mathcal{H} \\ \dot{\eta} = \text{Ad}_{g_-}^{G*} \Pi_{\mathfrak{g}_+}^* \text{Ad}_{g_-^{-1}}^{G*} \left(\text{ad}_{\text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta\mathcal{H}}^* \eta - g\mathbf{d}\mathcal{H} \right) \end{cases} \quad (9)$$

In terms of the components in G_{\pm} and \mathfrak{g}_{\pm}^* they means

$$\begin{cases} g_+^{-1}\dot{g}_+ = \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta\mathcal{H} \\ \dot{g}_- = 0 \\ \dot{\eta}_+ = \text{Ad}_{g_-}^{G*} \Pi_{\mathfrak{g}_+}^* \text{Ad}_{g_-^{-1}}^{G*} \left(\text{ad}_{\text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \delta\mathcal{H}}^* \eta - g\mathbf{d}\mathcal{H} \right) \\ \dot{\eta}_- = 0 \end{cases}$$

We shall be concerned with a dynamical system ruled by a Hamilton function of the type

$$\mathcal{H} = \mathbf{h} \circ J_B^{L\wedge} \quad / \quad \mathbf{h} : \mathfrak{g}^{\wedge*} \longrightarrow \mathbb{R}$$

with $J_B^{L\wedge}$ being the momentum map associated with the centrally extended left translation on (T^*G, ω_{\circ}) given in eq. (6). Despite it is apparently of collective type, actually it is not the case because $J_B^{L\wedge}$ is fails in to be a momentum map for (T^*G, ω_c) , which lacks of left invariance symmetry.

However, it is a truly collective system on $\mathcal{N}(g_-, \eta_-)$, for $(g_-, \eta_-) \in \ker C \times \text{Char}(\mathfrak{g}_-)$, where the centrally extended left symmetry is restored. In these fibers, the dynamics acquires interesting properties which we briefly recall now [10]. The derivative of some function \mathcal{F} along the flux of the hamiltonian vector field (5) is

$$\dot{\mathcal{F}} = \mathbf{L}_{V_{\mathcal{H}}} \mathcal{F} = \{\mathcal{F}, \mathcal{H}\}_c^D \quad (10)$$

Let us denote by $\varphi_{(g,\eta)} : G^{\wedge} \longrightarrow \mathcal{N}(g_-, \eta_-)$ the orbit map associated with the action (8) such that

$$\varphi_{(g,\eta)}(h, a) = \mathbf{d}((h, a), (g, \eta))$$

then, its differential at the neutral element of G^\wedge is the map $\varphi_{(g,\eta)*} : \mathfrak{g}^\wedge \longrightarrow T_{(g,\eta)}\mathcal{N}(g_-, \eta_-)$ gives rise to the infinitesimal generator $V_{(X,a)}^\mathcal{N}$ on $\mathcal{N}(g_-, \eta_-)$ defined in (7), then

$$\varphi_{(g,\eta)*}(X, a) := V_{(X,a)}^\mathcal{N}(g, \eta)$$

The Hamiltonian vector field associated to $\mathcal{H} = \mathfrak{h} \circ J_B^{L^\wedge}$ is then

$$V_{\mathcal{H}}(g, \eta) = \varphi_{(g,\eta)*} \mathcal{L}_{\mathfrak{h}}(J_B^{L^\wedge}(g, \eta))$$

where $\mathcal{L}_{\mathfrak{h}} : \mathfrak{g}^{\wedge*} \longrightarrow \mathfrak{g}^\wedge$ is defined as

$$\langle \xi, \mathcal{L}_{\mathfrak{h}}(\eta, b) \rangle := \left. \frac{d}{dt} \mathfrak{h}(\eta + t\xi, b) \right|_{t=0}$$

meaning that $V_{\mathcal{H}}(g, \eta)$ coincides with the infinitesimal generator associated with $\mathcal{L}_{\mathfrak{h}}(J_B^{L^\wedge}(g, \eta)) \in \mathfrak{g}^\wedge$. So, (10) is

$$\dot{\mathcal{F}} = \left\{ \mathcal{F}, j_{\mathcal{L}_{\mathfrak{h}} \circ J_B^{L^\wedge}}^D \right\}_c$$

In this way, at least locally, the integral curves of the hamiltonian vector field are orbits $\mathbf{d}((h(t), a), (g, \eta))$ of a curve $\gamma : \mathbb{R} \longrightarrow G / t \xrightarrow{\gamma} h(t)$. By the way, this curve is the same that solves the problem

$$\frac{d}{dt}(\eta, b) = \text{ad}_{\mathcal{L}_{\mathfrak{h}}(\eta, b)}^{\mathfrak{g}^{\wedge*}}(\eta, b)$$

which is equivalent to this one

$$\begin{cases} (\eta, b)(t) = \text{Ad}_{(h(t), a)}^{G^{\wedge*}}(\eta, b)(t_0) \\ (h(t), a)^{-1} \frac{d}{dt}(h(t), a) = \mathcal{L}_{\mathfrak{h}}(\eta, b) \end{cases}$$

Thus, the dynamical system on $\mathcal{N}(g_-, \eta_-)$ is replicated on the some coadjoint orbit in $\mathfrak{g}^{\wedge*}$, and the Ad-equivariance for the momentum maps allows to translate orbits in $\mathfrak{g}^{\wedge*}$ into orbits in $\mathcal{N}(g_-, \eta_-)$.

Part II

From WZNW type model to the Poisson-Lie σ -model

As the main application of the results of the first part, we study a hamiltonian systems which turns into collective type on the phase subspaces $\mathcal{N}(g_-, \eta_-)$, for $(g_-, \eta_-) \in \ker C \times \text{Char}(\mathfrak{g}_-)$. By deriving the Hamilton equations, we retrieve the Lagrange version of the model, showing that it corresponds to a generalization of the so called *Poisson-Lie σ -model*.

6 A collective Hamilton function on $\mathcal{N}(g_-, \eta_-)$

Motivated by the previous observation, we now propose to study a hamiltonian system ruled by the Hamilton function

$$\mathcal{H}(g, \eta) = \frac{1}{2} (\psi(J_B^{L^\wedge}(g, \eta)), \mathcal{E}\psi(J_B^{L^\wedge}(g, \eta)))_{\mathfrak{g}^\wedge} \quad (11)$$

where $J_B^{L\wedge}(g, \eta)$ is the centrally extended momentum map associated with left translations in the phase space (T^*G, ω_\circ) , introduced in eq. (6). The linear operator $\mathcal{E} : \mathfrak{g} \rightarrow \mathfrak{g}$ is idempotent $\mathcal{E}^2 = Id$, and breaks the Ad-invariance of the bilinear form. For further issues we introduce the operator $\mathcal{E}_g : \mathfrak{g} \rightarrow \mathfrak{g}$.

Definition: Let the operator $\mathcal{E}_g : \mathfrak{g} \rightarrow \mathfrak{g}$ be defined as

$$\mathcal{E}_g = \text{Ad}_{g^{-1}}^G \mathcal{E} \text{Ad}_g^G = \begin{pmatrix} -\mathcal{G}_g^{-1} \mathcal{B}_g & \mathcal{G}_g^{-1} \\ \mathcal{G}_g - \mathcal{B}_g \mathcal{G}_g^{-1} \mathcal{B}_g & \mathcal{B}_g \mathcal{G}_g^{-1} \end{pmatrix}$$

where

$$\begin{cases} \mathcal{G}_g = (\Pi_{\mathfrak{g}_+} \mathcal{E}_g \Pi_{\mathfrak{g}_-})^{-1} : \mathfrak{g}_+ \longrightarrow \mathfrak{g}_- / \langle \mathcal{G}_g X, Y \rangle = \langle X, \mathcal{G}_g Y \rangle \\ \mathcal{B}_g = -\mathcal{G}_g \circ \Pi_{\mathfrak{g}_+} \mathcal{E}_g \Pi_{\mathfrak{g}_+} : \mathfrak{g}_+ \longrightarrow \mathfrak{g}_- / \langle \mathcal{B}_g X, Y \rangle = -\langle X, \mathcal{B}_g Y \rangle \end{cases}$$

which makes \mathcal{E}_g symmetric

$$((Y, \xi), \mathcal{E}_g(X, \eta))_{L\mathcal{D}} = (\mathcal{E}_g(Y, \xi), (X, \eta))_{L\mathcal{D}}$$

and idempotent

$$\mathcal{E}_g \circ \mathcal{E}_g = Id_{L\mathcal{D}}$$

It gives rise to a decomposition of \mathfrak{g} in two eigenvalue subspaces

$$\mathcal{E}^\pm(g) = \{X \in \mathfrak{g} / \mathcal{E}_g X = \pm X\}$$

such that $\mathfrak{g} = \mathcal{E}^+ \oplus \mathcal{E}^-$. Therefore, if X is in $\mathcal{E}^\pm(g)$ it can be written as

$$X = (X_+, (\mathcal{B}_g \pm \mathcal{G}_g) X_+)$$

or, alternatively, as

$$X = ((\mathcal{B}_g \pm \mathcal{G}_g)^{-1} X_-, X_-)$$

Observe that if $X \in \mathcal{E}^\pm(g)$, then $\text{Ad}_{g_+}^G X \in \mathcal{E}^\pm(e)$.

Coming back to the Hamilton function on $G \times \mathfrak{g}^*$, we write

$$\mathcal{H}(g, \eta) = \frac{1}{2} \left(\bar{\psi} \left(\text{Ad}_{g^{-1}}^{G*} \eta + C(g) \right), \mathcal{E} \bar{\psi} \left(\text{Ad}_{g^{-1}}^{G*} \eta + C(g) \right) \right)_{\mathfrak{g}}$$

and its differential $d\mathcal{H} = (\mathbf{d}\mathcal{H}, \delta\mathcal{H})$ is

$$\begin{cases} \mathbf{d}\mathcal{H} = g^{-1} \psi \left[\bar{\psi} (\eta - C(g^{-1})), \mathcal{E}_g \bar{\psi} (\eta - C(g^{-1})) \right] \\ \quad + g^{-1} \text{Ad}_g^{G*} \hat{c} \left(\mathcal{E} \text{Ad}_g^G \bar{\psi} (\eta - C(g^{-1})) \right) \\ \delta\mathcal{H} = \mathcal{E}_g \bar{\psi} (\eta - C(g^{-1})) \end{cases} \quad (12)$$

The Hamilton equations in $G \times \mathfrak{g}^*$ are obtained from the hamiltonian vector field (2) giving

$$\begin{cases} g^{-1} \dot{g} = \mathcal{E}_g \bar{\psi} (\eta - C(g^{-1})) \\ \dot{\eta} = \text{ad}_{\mathcal{E}_g \bar{\psi} (\eta - C(g^{-1}))}^{\mathfrak{g}*} C(g^{-1}) \end{cases}$$

It is worth to remark here that, in retrieving the Lagrange equation, we get the second order equation

$$\frac{\partial}{\partial t} (\mathcal{E}_g (g^{-1} \dot{g}) + \bar{\psi} (C(g^{-1}))) = [g^{-1} \dot{g}, \bar{\psi} (C(g^{-1}))]$$

where the Lie algebra bracket in the rhs is a manifestation of the topological WZ term.

6.1 The Hamilton equation in $\mathcal{N}(g_-, \eta_-)$

The Hamilton equation in $\mathcal{N}(g_-, \eta_-)$ where given in (9). Since we shall be concerned with loop groups, from now on we assume that the cocycle C restricts to the factors as the map $C : G_{\pm} \rightarrow \mathfrak{g}_{\mp}^*$.

Then, replacing $\mathbf{d}\mathcal{H}, \delta\mathcal{H}$ in those Hamilton equation by the ones given in (12) we get the Hamilton equations for the Hamilton function (11). Before to write them in terms of the components $G_{\pm} \times \mathfrak{g}_{\pm}^*$, we verify that they are indeed generated by an infinitesimal generator of the action (8). In fact, after replacing the differential (12) in the equations (9), we can handle them to the form

$$\begin{cases} g^{-1}\dot{g} = \text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_+^{-1}}^G \left(\text{Ad}_g^G \mathcal{E}_g \bar{\psi} (\eta - C(g^{-1})) \right) \\ \dot{\eta} = -\text{Ad}_{g_-}^{G*} \Pi_{\mathfrak{g}_+}^* \text{Ad}_{g_-^{-1}}^{G*} \left(\text{ad}_{\text{Ad}_{g_-^{-1}}^G \Pi_{\mathfrak{g}_+} \text{Ad}_{g_+^{-1}}^G (\text{Ad}_g^G \mathcal{E}_g \bar{\psi} (\eta - C(g^{-1})))}^{\mathfrak{g}^*} \eta \right) \\ \quad - \text{Ad}_{g_-}^{G*} \Pi_{\mathfrak{g}_+}^* \text{Ad}_{g_-^{-1}}^{G*} \left(\hat{c} \left(\text{Ad}_{g_-^{-1}}^G \left(\text{Ad}_g^G \mathcal{E}_g \bar{\psi} (\eta - C(g^{-1})) \right) \right) \right) \end{cases}$$

verifying that

$$(\dot{g}, \dot{\eta}) = V_{(\text{Ad}_g^G \mathcal{E}_g \bar{\psi} (\eta - C(g^{-1})), a)}^{\mathcal{N}}(g, \eta)$$

for $V_{(X,a)}^{\mathcal{N}}$ defined in (7), as we expected from the collective character of the dynamics in $\mathcal{N}(g_-, \eta_-)$.

Now, let us work out the dynamics in $\mathcal{N}(g_-, \eta_-)$ in terms of its components $G_{\pm} \times \mathfrak{g}_{\pm}^*$ by writing $g = g_+ g_-$ and $\eta = \eta_+ + \eta_-$. Of course, (g_-, η_-) remains frozen, while (g_+, η_+) evolves in $\mathcal{N}(g_-, \eta_-)$ ruled by the equations:

$$\begin{cases} g_+^{-1} \dot{g}_+ = \Pi_{\mathfrak{g}_+} \text{Ad}_{g_+^{-1}}^G \mathcal{E} \text{Ad}_g^G \bar{\psi} (\eta - C(g^{-1})) \\ \dot{\eta} = -\text{Ad}_{g_-}^{G*} \Pi_{\mathfrak{g}_+}^* \psi \left[\text{Ad}_{g_-}^G \bar{\psi} (\eta - C(g^{-1})), \Pi_{\mathfrak{g}_-} \text{Ad}_{g_-}^G \mathcal{E}_g \bar{\psi} (\eta - C(g^{-1})) \right] \\ \quad + \text{Ad}_{g_-}^{G*} \Pi_{\mathfrak{g}_+}^* \psi \left[\text{Ad}_{g_-}^G \bar{\psi} (C(g^{-1})), \Pi_{\mathfrak{g}_+} \text{Ad}_{g_-}^G \mathcal{E}_g \bar{\psi} (\eta - C(g^{-1})) \right] \\ \quad - \text{Ad}_{g_-}^{G*} \Pi_{\mathfrak{g}_+}^* \text{Ad}_{g_+}^{G*} \hat{c} \left(\mathcal{E} \text{Ad}_g^G \bar{\psi} (\eta - C(g^{-1})) \right) \end{cases}$$

Here we observe that, by taking $(g_-, \eta_-) \in \ker C \times \text{Char}(\mathfrak{g}_-)$ it happens that

$$C(g^{-1}) = C(g_-^{-1} g_+^{-1}) = \text{Ad}_{g_-}^{G*} C(g_+^{-1})$$

Then, having since for $\eta_{\pm} \in \mathfrak{g}_{\pm}^*$ and $g_{\mp} \in G_{\mp}$, $\text{Ad}_{g_{\mp}}^{G*} \eta_{\pm} \in \mathfrak{g}_{\pm}^*$, which also means that $\text{ad}_{X_{\mp}}^{\mathfrak{g}^*} \eta_{\pm} = \Pi_{\mathfrak{g}_+}^* \text{ad}_{X_{\mp}}^{\mathfrak{g}^*} \eta_{\pm}$. Again, using these facts and the properties of the different cocycles, we arrive to

$$\begin{cases} g_+^{-1} \dot{g}_+ = \Pi_{\mathfrak{g}_+} \mathcal{E}_{g_+} \bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \eta - C(g_+^{-1}) \right) \\ \text{Ad}_{g_-^{-1}}^{G*} \dot{\eta} = -\Pi_{\mathfrak{g}_+}^* \psi \left[\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \eta \right), \Pi_{\mathfrak{g}_-} \mathcal{E}_{g_+} \bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \eta - C(g_+^{-1}) \right) \right] \\ \quad - \Pi_{\mathfrak{g}_+}^* \hat{c} \left(\Pi_{\mathfrak{g}_-} \mathcal{E}_{g_+} \bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \eta - C(g_+^{-1}) \right) \right) \end{cases}$$

7 The Lagrange equation

We now proceed to retrieve the Lagrange equation on $\mathcal{N}(g_-, \eta_-)$ by replacing the first Hamilton equation into the second one. Therefore, since η_- is a character of \mathfrak{g}_- ,

$$\Pi_{\mathfrak{g}_-}^* \text{Ad}_{g_-^{-1}}^{G*} \eta_- = \text{Ad}_{g_-^{-1}}^{G*} \eta_- = \eta_-$$

the first Hamilton equation becomes in

$$g_+^{-1} \dot{g}_+ = \mathcal{G}_{g_+}^{-1} \bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \eta_+ \right) + \mathcal{G}_{g_+}^{-1} \bar{\psi} \left(\Pi_{\mathfrak{g}_+^*} \text{Ad}_{g_-^{-1}}^{G*} \eta_- \right) + \mathcal{G}_{g_+}^{-1} \mathcal{B}_{g_+} \bar{\psi} \left(C(g_+^{-1}) - \eta_- \right)$$

then

$$\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \eta_+ \right) = \mathcal{G}_{g_+} g_+^{-1} \dot{g}_+ - \mathcal{B}_{g_+} \bar{\psi} \left(C(g_+^{-1}) \right) + \left(\mathcal{B}_{g_+} - \Pi_{\mathfrak{g}_-} \text{Ad}_{g_-}^G \right) \bar{\psi}(\eta_-) \quad (13)$$

The second Hamilton equation can be handled to

$$\begin{aligned} & \bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \dot{\eta} \right) \\ = & -\Pi_{\mathfrak{g}_-} \left[\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \eta \right), \left(\mathcal{G}_{g_+} - \mathcal{B}_{g_+} (\mathcal{G}_{g_+})^{-1} \mathcal{B}_{g_+} \right) \Pi_{\mathfrak{g}_+} \bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \eta - C(g_+^{-1}) \right) \right] \\ & -\Pi_{\mathfrak{g}_-} \left[\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \eta \right), \mathcal{B}_{g_+} (\mathcal{G}_{g_+})^{-1} \Pi_{\mathfrak{g}_-} \bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \eta - C(g_+^{-1}) \right) \right] \\ & -\bar{\psi} \left(\hat{c} \left(\left(\mathcal{G}_{g_+} - \mathcal{B}_{g_+} (\mathcal{G}_{g_+})^{-1} \mathcal{B}_{g_+} \right) \Pi_{\mathfrak{g}_+} \bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \eta - C(g_+^{-1}) \right) \right) \right) \\ & -\bar{\psi} \left(\hat{c} \left(\mathcal{B}_{g_+} (\mathcal{G}_{g_+})^{-1} \Pi_{\mathfrak{g}_-} \bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \eta - C(g_+^{-1}) \right) \right) \right) \end{aligned}$$

and by substituting $\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \eta_+ \right)$ as given in eq. (13) we get

$$\begin{aligned} & \frac{d}{dt} \left(\mathcal{G}_{g_+} g_+^{-1} \dot{g}_+ - \mathcal{B}_{g_+} \bar{\psi} \left(C(g_+^{-1}) \right) + \mathcal{B}_{g_+} \bar{\psi}(\eta_-) \right) \\ = & -\Pi_{\mathfrak{g}_-} \left[\mathcal{G}_{g_+} g_+^{-1} \dot{g}_+ - \mathcal{B}_{g_+} \bar{\psi} \left(C(g_+^{-1}) - \eta_- \right), \mathcal{B}_{g_+} g_+^{-1} \dot{g}_+ - \mathcal{G}_{g_+} \bar{\psi} \left(C(g_+^{-1}) - \eta_- \right) \right] \\ & -\Pi_{\mathfrak{g}_-} \left[\bar{\psi}(\eta_-), \mathcal{B}_{g_+} g_+^{-1} \dot{g}_+ - \mathcal{G}_{g_+} \bar{\psi} \left(C(g_+^{-1}) - \eta_- \right) \right] \\ & -\bar{\psi} \left(\hat{c} \left(\mathcal{B}_{g_+} g_+^{-1} \dot{g}_+ - \mathcal{G}_{g_+} \bar{\psi} \left(C(g_+^{-1}) - \eta_- \right) \right) \right) \end{aligned} \quad (14)$$

On the fiber $\mathcal{N}(e, 0) \cong G_+ \times \mathfrak{g}_+^*$ it reduces to

$$\begin{aligned} & \frac{d}{dt} \psi \left(\mathcal{G}_{g_+} g_+^{-1} \dot{g}_+ - \mathcal{B}_{g_+} \bar{\psi} \left(C(g_+^{-1}) \right) \right) + \hat{c} \left(\mathcal{B}_{g_+} g_+^{-1} \dot{g}_+ - \mathcal{G}_{g_+} \bar{\psi} \left(C(g_+^{-1}) \right) \right) \\ = & -\psi \left[\mathcal{G}_{g_+} g_+^{-1} \dot{g}_+ - \mathcal{B}_{g_+} \bar{\psi} \left(C(g_+^{-1}) \right), \mathcal{B}_{g_+} g_+^{-1} \dot{g}_+ - \mathcal{G}_{g_+} \bar{\psi} \left(C(g_+^{-1}) \right) \right] \end{aligned}$$

8 The Lagrangian function on $\mathcal{N}(g_-, \eta_-)$

Now, we built up the Lagrangian version on the fibers $\mathcal{N}(g_-, \eta_-)$, which in turn are symplectic submanifolds equipped with the restriction of the symplectic form on $G \times \mathfrak{g}^*$. The Lagrangian relates with the Hamilton function by

$$dL = \Theta - dH$$

provided there exist the 1-form Θ such satisfying

$$\omega_c|_{\mathcal{N}(g_-, \eta_-)} = -d\Theta$$

In the current case, it is supplied by the following proposition.

Proposition: *The restriction of the symplectic form ω_c to $\mathcal{N}(g_-, \eta_-)$ is an exact form such that*

$$\omega_c|_{\mathcal{N}(g_-, \eta_-)} = -d\Theta$$

with Θ being the restriction of the canonical 1-form of $G \times \mathfrak{g}^*$ to $\mathcal{N}(g_-, \eta_-)$

$$\left\langle \Theta, \left(g_+ \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \psi(X_+) \right) \right) g_-, \xi_+ \right) \right\rangle_{(g, \eta)} := \left\langle \eta, g_+ \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^{G*} \psi(X_+) \right) \right) g_- \right\rangle$$

Proof: By the isotropic character of the subspaces \mathfrak{g}_\pm under the cocycle c , the restriction of ω_c reduces to

$$\begin{aligned} & \left\langle \omega, \left(g_+ \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(X_+) \right) \right) g_-, \xi_+ \right) \otimes \left(g_+ \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(Y_+) \right) \right) g_-, \lambda_+ \right) \right\rangle_{(g, \eta)} \\ &= - \left\langle \xi_+, \text{Ad}_{g_-^{-1}}^G \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(Y_+) \right) \right) \right\rangle + \left\langle \lambda_+, \text{Ad}_{g_-^{-1}}^G \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(X_+) \right) \right) \right\rangle \\ & \quad + \left\langle \eta, \left[\text{Ad}_{g_-^{-1}}^G \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(X_+) \right) \right), \text{Ad}_{g_-^{-1}}^G \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(Y_+) \right) \right) \right] \right\rangle \end{aligned}$$

Let us define it as $\Theta \in T^*\mathcal{N}(g_-, \eta_-)$ such that

$$\left\langle \Theta, \left(g_+ \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(X_+) \right) \right) g_-, \xi_+ \right) \right\rangle_{(g, \eta)} := \left\langle \eta, g_+ \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(X_+) \right) \right) g_- \right\rangle$$

then

$$\begin{aligned} & \left\langle d\Theta, \left(g_+ \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(X_+) \right) \right) g_-, \xi_+ \right) \otimes \left(g_+ \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(Y_+) \right) \right) g_-, \lambda_+ \right) \right\rangle_{(g, \eta)} \\ &= \left\langle \xi_+, g_+ \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(Y_+) \right) \right) g_- \right\rangle - \left\langle \lambda_+, g_+ \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(X_+) \right) \right) g_- \right\rangle \\ & \quad - \left\langle \eta, \left[g_+ \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(X_+) \right) \right) g_-, g_+ \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^* \psi(Y_+) \right) \right) g_- \right] \right\rangle \end{aligned}$$

showing that $\omega_c|_{\mathcal{N}(g_-, \eta_-)} = -d\Theta$ as stated. ■

Therefore, the Lagrangian function on $\mathcal{N}(g_-, \eta_-)$ is

$$L_{\mathcal{N}(g_-, \eta_-)}(g, \dot{g}) = \langle \eta, g^{-1} \dot{g} \rangle - \mathcal{H}(g, \eta)|_{\mathcal{N}(g_-, \eta_-)}$$

where $\mathcal{H}(g, \eta)|_{\mathcal{N}(g_-, \eta_-)}$ is the restriction of $\mathcal{H}(g, \eta)$ to $\mathcal{N}(g_-, \eta_-)$, with

$$\mathcal{H}(g, \eta) = \frac{1}{2} \left(\bar{\psi} \left(J_B^{L^\wedge}(g, \eta) \right), \mathcal{E} \bar{\psi} \left(J_B^{L^\wedge}(g, \eta) \right) \right)_{\mathfrak{g}^\wedge}$$

where, having in mind that $(g_-, \eta_-) \in \ker C \times \text{Char}(\mathfrak{g}_-)$,

$$J_B^{L^\wedge}(g, \eta)|_{\mathcal{N}(g_-, \eta_-)} = \left(\text{Ad}_{g_-^{-1}}^G \eta + C(g_+), 1 \right)$$

Then

$$\begin{aligned} H(g_+, \eta_+) & : = \mathcal{H}(g, \eta)|_{\mathcal{N}(g_-, \eta_-)} \\ &= \frac{1}{2} \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^G \eta + C(g_+) \right), \mathcal{E} \bar{\psi} \left(\text{Ad}_{g_-^{-1}}^G \eta + C(g_+) \right) \right)_{\mathfrak{g}^\wedge} \end{aligned} \tag{15}$$

On the other side, on $\mathcal{N}(g_-, \eta_-)$ we have that

$$g^{-1} \dot{g} = g_-^{-1} g_+^{-1} \dot{g}_+ g_- = \text{Ad}_{g_-^{-1}}^G (g_+^{-1} \dot{g}_+)$$

therefore

$$\begin{aligned} & L_{\mathcal{N}}(g, \dot{g}) \\ &= \left\langle \eta, \text{Ad}_{g_-^{-1}}^G (g_+^{-1} \dot{g}_+) \right\rangle - \frac{1}{2} \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^G \eta + C(g_+) \right), \mathcal{E} \bar{\psi} \left(\text{Ad}_{g_-^{-1}}^G \eta + C(g_+) \right) \right)_{\mathfrak{g}} \end{aligned}$$

that is equivalent to

$$\begin{aligned} & L_{\mathcal{N}}(g, \dot{g}) \\ &= \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^G \eta \right), g_+^{-1} \dot{g}_+ \right)_{\mathfrak{g}} - \frac{1}{2} \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^G \eta \right), \mathcal{E}_{g_+} \bar{\psi} \left(\text{Ad}_{g_-^{-1}}^G \eta \right) \right)_{\mathfrak{g}} \\ & \quad + \left(\bar{\psi} \left(\text{Ad}_{g_-^{-1}}^G \eta \right), \mathcal{E}_{g_+} \bar{\psi} \left(C(g_+^{-1}) \right) \right)_{\mathfrak{g}} - \frac{1}{2} \left(\bar{\psi} \left(C(g_+^{-1}) \right), \mathcal{E}_{g_+} \bar{\psi} \left(C(g_+^{-1}) \right) \right)_{\mathfrak{g}} \end{aligned}$$

By replacing the fiber coordinate η_+ as a function of the velocity, as it was obtained from the first Hamilton equation in eq. (13),

$$Ad_{g_+^{-1}}^{G^*} \eta_+ = \psi \left(\mathcal{G}_{g_+} g_+^{-1} \dot{g}_+ - \mathcal{B}_{g_+} \bar{\psi} (C(g_+^{-1})) + \left(\mathcal{B}_{g_+} - \Pi_{\mathfrak{g}_-} Ad_{g_+}^G \right) \bar{\psi}(\eta_-) \right)$$

we get

$$\begin{aligned} L_{\mathcal{N}}(g, \dot{g}) &= \frac{1}{2} (\mathcal{G}_{g_+} g_+^{-1} \dot{g}_+, g_+^{-1} \dot{g}_+)_{\mathfrak{g}} - (g_+^{-1} \dot{g}_+, \mathcal{B}_{g_+} \bar{\psi} (C(g_+^{-1}) - \eta_-))_{\mathfrak{g}} \\ &\quad - \frac{1}{2} (\bar{\psi} (C(g_+^{-1}) - \eta_-), \mathcal{G}_{g_+} \bar{\psi} (C(g_+^{-1}) - \eta_-))_{\mathfrak{g}} \end{aligned} \quad (16)$$

Finally, it can be written also as the Poisson-Lie σ -model:

$$L_{\mathcal{N}}(g, \dot{g}) = \frac{1}{2} \left(\mathcal{R}_{g_+}^+ (g_+^{-1} \dot{g}_+ - \bar{\psi} (C(g_+^{-1}) - \eta_-)), (g_+^{-1} \dot{g}_+ + \bar{\psi} (C(g_+^{-1}) - \eta_-)) \right)_{\mathfrak{g}}$$

where $\mathcal{R}_{g_+}^{\pm} := \mathcal{B}_{g_+} \pm \mathcal{G}_{g_+}$.

Thus, we have shown that the collective hamiltonian system (15) is the phase space version of the lagrangian system introduced in the context Poisson-Lie T-duality by [12]. Amazingly, it arises through the Dirac restriction method of the larger phase $G \times \mathfrak{g}^*$ of a noncollective nonAd-invariant hamiltonian function.

8.1 Centrally extended loop algebras

Let consider \mathfrak{g} as being the loop algebra $\mathfrak{g} = L\mathfrak{h}$ for some Lie algebra \mathfrak{h} equipped with the bilinear form

$$(X, Y)_{\mathfrak{g}} = \frac{1}{2\pi} \int_{S^1} (X(s), Y(s))_{\mathfrak{h}} ds$$

Then we define the coadjoint cocycle $C_k : H \rightarrow \mathfrak{h}^*$,

$$C_k(g) = k\psi(g'g^{-1}) \quad (17)$$

which satisfy

$$C_k(gh) = Ad_{g^{-1}}^{G^*} C_k(h) + C_k(g)$$

It extend to a cocycle on $G = LH$ by the identification of the dual \mathfrak{g}^* through the bilinear form above defined, in such a way that

$$\langle C_k(g), Y \rangle = \frac{k}{2\pi} \int_{S^1} (g'g^{-1}, Y)_{\mathfrak{h}} ds$$

so we define $c_k : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R}$ as

$$c_k(X, Y) = \frac{k}{2\pi} \int_{S^1} (X(s), Y'(s))_{\mathfrak{h}} ds$$

with the map $\hat{c}_k : \mathfrak{g} \longrightarrow \mathfrak{g}^*$

$$\hat{c}_k(X) = -k\psi(X') \quad (18)$$

Let us substitute these cocycle (17) and (18) in the Lagrange equation (14), to obtain

$$\begin{aligned} &\frac{\partial}{\partial t} (\mathcal{G}_{g_+} g_+^{-1} \dot{g}_+ + k\mathcal{B}_{g_+} g_+^{-1} g'_+ + \mathcal{B}_{g_+} \bar{\psi}(\eta_-) + \bar{\psi}(\eta_-)) \\ &- k \frac{\partial}{\partial x} (\mathcal{B}_{g_+} g_+^{-1} \dot{g}_+ + k\mathcal{G}_{g_+} g_+^{-1} g'_+ + \mathcal{G}_{g_+} \bar{\psi}(\eta_-)) \\ &= \Pi_{\mathfrak{g}_-} [\mathcal{B}_{g_+} g_+^{-1} \dot{g}_+ + \mathcal{G}_{g_+} (kg_+^{-1} g'_+ + \bar{\psi}(\eta_-)), \mathcal{G}_{g_+} g_+^{-1} \dot{g}_+ + \mathcal{B}_{g_+} (kg_+^{-1} g'_+ + \bar{\psi}(\eta_-)) + \bar{\psi}(\eta_-)] \end{aligned}$$

In this case, and introducing $\partial_{\pm} = \frac{\partial}{\partial t} \pm k \frac{\partial}{\partial x}$, the Lagrangian density (16) turns in

$$\mathcal{L}_{\mathcal{N}}(g_+) = \frac{1}{2} (\mathcal{R}_{g_+} (g_+^{-1} \partial_+ g_+ + \bar{\psi}(\eta_-)), (g_+^{-1} \partial_- g_+ - \bar{\psi}(\eta_-)))_{\mathfrak{g}} \quad (19)$$

In order to eliminate the g_+ -dependence in the operator \mathcal{R}_{g_+} , leaving a purely world sheet depending one, we use the decomposition of the Lie algebra \mathfrak{g} as the direct sum of the eigenspaces of the operator \mathcal{E}_{g_+} following reference [14]. Since $((\mathcal{R}_e^{\pm})^{-1} X_-, X_-) \in \mathcal{E}^{\pm}(e)$, $\text{Ad}_{g_+}^G ((\mathcal{R}_e^{\pm})^{-1} X_-, X_-) \in \mathcal{E}^{\pm}(g)$ which implies the relation

$$\text{Ad}_{g_+}^G (\mathcal{R}_g^{\pm})^{-1} \Pi_{\mathfrak{g}-} \text{Ad}_{g_+}^G X_- = (\mathcal{R}_e^{\pm})^{-1} X_- + \Pi_{\mathfrak{g}+} \text{Ad}_{g_+}^G \Pi_{\mathfrak{g}+} \text{Ad}_{g_+}^G X_-$$

By using that the right translated Poisson-Lie bivector on G_+ , $\pi_+^R : G_+ \longrightarrow \mathfrak{g}_+ \otimes \mathfrak{g}_+$, is defined from the relation [13]

$$\langle \psi(X'_-) \otimes \psi(X''_-), \pi_+^R(g_+) \rangle := (\Pi_- \text{Ad}_{g_+}^G X'_-, \Pi_+ \text{Ad}_{g_+}^G X''_-)_{\mathfrak{g}}$$

and regarding it as the linear map $\pi_+^R(g_+) : \mathfrak{g}_- \longrightarrow \mathfrak{g}_+$ such that

$$(\pi_+^R(g_+) X_-, Y_-)_{\mathfrak{g}} := \langle \psi(Y_-) \otimes \psi(X_-), \pi_+^R(g_+) \rangle$$

$\forall Y_- \in \mathfrak{g}_-$, we identify

$$\pi_+^R(g_+) = -\Pi_{\mathfrak{g}+} \text{Ad}_{g_+}^G \Pi_{\mathfrak{g}+} \text{Ad}_{g_+}^G \Pi_-$$

So, the above relation turns in

$$\text{Ad}_{g_+}^G (\mathcal{R}_g^{\pm})^{-1} \Pi_{\mathfrak{g}-} \text{Ad}_{g_+}^G \Pi_{\mathfrak{g}-} = ((\mathcal{R}_e^{\pm})^{-1} - \pi_+^R(g_+)) \Pi_{\mathfrak{g}-}$$

which allows to substitute can be substitute \mathcal{R}_g^{\pm} into the Lagrangian density (19) to get

$$\mathcal{L}_{\mathcal{N}}(g) = \frac{1}{2} \left(((\mathcal{R}_e^{\pm})^{-1} - \pi_+^R(g_+))^{-1} (g_+^{-1} \partial_+ g_+ + \bar{\psi}(\eta_-)), (g_+^{-1} \partial_- g_+ - \bar{\psi}(\eta_-)) \right)_{\mathfrak{g}}$$

that coincides with the T-dual sigma model on the target G_+ introduced by Klimcik and Severa [12].

Conclusions

We have studied the restriction to a family of submanifolds of the second class constraint type in the cotangent bundle of a double Poisson-Lie group, equipped with a cocycle extended symplectic form. For a given hamiltonian system on this phase space, it is this extension which give rise to the WZ topological term in the corresponding action. Then, we build up the corresponding Dirac brackets showing that, for the usual 2-cocycle in loop groups, it exhibit no contribution coming from it. It is in this sense that we say that a $WZNW$ phase space restricts to a σ -model one. Looking for symmetries on the constrained phase spaces, we worked out the left translation action despite it is manifestly not a symmetry in the whole space because of the 2-cocycle is defined on right invariant vector fields. However, we have shown that in the fiber spaces on points $(g_-, \eta_-) \in \ker C \times \text{Char}(\mathfrak{g}_-)$, the left translation momentum function close a Lie algebra under the corresponding Dirac bracket, so the left translation symmetry becomes restored.

This facts suggested to studied collective dynamical systems on these phase subspaces, which in some way resembles the situation in the hamiltonian approach WZNW model, see ref. [11], where the Marsden-Weinstein reduction procedure has a first stage of reduction in relation with the obvious right translation symmetry. However, there still remains a residual left translation symmetry which, once reduced, leads to the chiral modes solutions.

Thus we studied a hamiltonian system on $G \times \mathfrak{g}^*$ ruled by a Hamilton function of the momentum maps of left translation in the constrained submanifolds, regarded as functions on the whole space. This hamiltonian turns collective when restricted to $\mathcal{N}(g_-, \eta_-)$, and leads to nice dynamics flowing along orbits of some curves in the group G through the action found in (8). Amazingly, these restricted hamiltonian systems turn to be Poisson-Lie σ -models, whit some additional terms involving the parameter η_- of the fiber. In summary, starting from a WZNW type model, we arrived through a restriction procedure implemented by the Dirac method to the Poisson-Lie σ -models.

Most of the issues developed here are much involved with the hamiltonian framework for the Poisson-Lie T-duality, as it is clear from the restriction of the left translation symmetry to the special fibers. The T-dual side can be obtained, with some subtleties, by considering the fibration $G \times \mathfrak{g}^* \longrightarrow G_+ \times \mathfrak{g}_+^*$ as starting point, and repeating the above construction.

Although the current work is purely classical, this formulation as a second class constrained systems allow to address the quantization in the scheme of functional integral as a constrained system defined on the whole $G \times \mathfrak{g}^*$.

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